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Nondispersive electromagnetic beams in plasmas

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Abstract. We prove that different modes of nondispersive electromagnetic beams can propagate in a stationary isotropic plasma. But, a stationary plasma in a uniform magnetic field may only support a mode at frequencies less than the angular cyclotron frequency.

PACS. 43.20.Bi Mathematical theory of wave propagation – 41.20.Jb Electromagnetic wave propagation; radiowave propagation – 52.35.Hr Electromagnetic waves (e.g., electron-cyclotron, Whistler, Bernstein, upper hybrid, lower hybrid)

1 Introduction

Some years ago, people became interested to transmit energy at large distances and a condition to reach this objective is the possibility to generate electromagnetic beams propagating with a minimum of dispersion. Gaussian beams at the output of powerful lasers were the first practical solution to this problem. But, it has been known for a long time [1] that free space supports electromagnetic beams with better performances, that is with less dispersion, and the quest of nondiffractive solutions to the wave and Maxwell equations, still flourishing [2–4] has given a great diversity of nondispersive fields [5,6] among which the Bessel beams discovered by Durnin [7] are the most attractive since they can be easily generated [8] for practical uses [9,10].

A natural question is whether nondispersive electromagnetic beams can propagate in media the most often experienced in physics and engineering as they do in free space. This possibility was discussed recently, in particular for propagation in nonhomogeneous media [11]. Now, plasmas are the normal state of matter in Nature and they occur more and more frequently in modern technology; all these reasons lead to investigate whether plasmas can support nondispersive waves, a question never considered till now. We prove here that different modes of nondispersive Bessel beams may exist in a stationary isotropic plasma. At the opposite, a plasma in a uniform magnetic field with a permittivity tensor presenting a tridiagonal symmetry can only support such a mode at frequencies less than the angular cyclotron frequency. We conjecture that this symmetry is the weakest making possible nondispersive solutions of Maxwell's equations (plane waves excepted) in anisotropic media.

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This paper is organized as follows: Section 2 is devoted to electromagnetic Bessel beams in a stationary plasma considered, provided that some approximations are imposed, as a linear isotropic medium and we show the existence of many different nondispersive modes. We prove in Section 3 that a stationary plasma in a steady uniform field with a tridiagonal permittivity tensor can only support a mode at frequencies less than the angular cyclotron frequency. Conclusive comments are given in Section 4.

2 Bessel beam propagation in a stationary isotropic plasma

The plasma will be considered as stationary and we assume that the ions play no role serving only to neutralize the dc-fields of electrons. The effect of collisions and of thermal velocities as well as the forces resulting from the magnetic component of the wave are neglected. Then, this plasma becomes a linear isotropic medium with Fourier transform $\varepsilon(\omega)$ of permittivity so that according to Maxwell's theory the electric field $\mathbf{E}(\mathbf{x},\omega)$ satisfies the following equations in which $\mathbf{D}(\mathbf{x},\omega)$ is the electric induction

$$\partial^j\partial_j E_k(\mathbf{x},\omega) - \partial_k\partial^j E_j(\mathbf{x},\omega) + \omega^2 c^{-2}\varepsilon(\omega)E_k(\mathbf{x},\omega) = 0 \qquad \mathbf{x} = (x,y,z) \quad (1)$$

$$\partial^j D_j(\mathbf{x},\omega) = \varepsilon(\omega)\partial^j E_j(\mathbf{x},\omega) = \rho(\mathbf{x},\omega). \quad (1a)$$

The subscripts j, k, take the values 1, 2, 3, corresponding to the coordinates x, y, z, and we use the convention summation on repeated indices, the permeability μ is assumed unity, ρ is the charge density and with the collisions neglected the permittivity $\varepsilon(\omega)$ is [12,13]

$$\varepsilon(\omega) = \varepsilon_0(1 - \omega_p^2\omega^{-2}) \quad (2)$$

where ε_0 is the free space permittivity and ω_p the plasma frequency.

The density $\rho = 0$ in a neutral plasma and equations (1, 1a) reduce to

$$\partial^j \partial_j E_k(\mathbf{x}, \omega) + \omega^2 c^{-2}\, \varepsilon(\omega) E_k(\mathbf{x}, \omega) = 0, \qquad \partial^j E_j(\mathbf{x}, \omega) = 0 \tag{3}$$

and these equations have travelling wave solutions for $\omega > \omega_p$ among which the elementary ones with amplitude A_j

$$E_j(\mathbf{x}, \omega) = A_j \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}), \quad |\mathbf{k}|^2 = \omega^2 c^{-2} \varepsilon_0 (1 - \omega_p^2 \omega^{-2}), \qquad k^j A_j = 0 \tag{4}$$

with the phase and group velocities

$$v_p = c\left[\varepsilon_0(1-\omega_p^2\omega^{-2})\right]^{-1/2}, \quad v_g = c\left[\varepsilon_0(1-\omega_p^2\omega^{-2})\right]^{1/2}. \tag{4a}$$

For $\omega < \omega_p$, the intrinsic impedance of the plasma is imaginary, so the magnetic field is out of phase with the electric field. In addition, the wave number k is also imaginary, and we have to deal with attenuated waves of the type $\exp(-kz)$ and $\exp(kz)$ respectively for propagation in the positive and negative z-direction with $k = \omega c\left[\varepsilon_0(\omega_p^2\omega^{-2}-1)\right]^{1/2}$.

But the wave equation (3) has also the solutions

$$E_j(\mathbf{x}, \omega) = \exp(-\mathrm{i}\beta z) \times \int_0^{2\pi} \mathrm{d}\phi A_j(\phi) \exp[\mathrm{i}\alpha(x\cos\phi + y\sin\phi)] \tag{5}$$

with

$$\alpha^2 + \beta^2 = \omega^2 c^{-2} \varepsilon(\omega) \tag{5a}$$

in which the amplitudes A_j must satisfy the condition imposed by the divergence equation (3)

$$\alpha[\cos\phi A_x(\phi) + \sin\phi A_y(\phi)] - \beta A_z(\phi) = 0. \tag{6}$$

Equation (6) has an the infinity of solutions among which the particular modes
TEM:

$$A_z(\phi) = 0, \quad A_x(\phi) = A(\phi)\sin\phi, \quad A_y(\phi) = -A(\phi)\cos\phi$$

TM:

$$A_x(\phi) = 0, \;\; A_y(\phi) = \alpha^{-1}\beta\cos\phi A_z(\phi); \qquad A_y(\phi) = 0, \;\; A_x(\phi) = \alpha^{-1}\beta\sin\phi A_z(\phi)$$

nontransverse:

$$A_x(\phi) = \cos\phi A(\phi), \quad A_y(\phi) = \sin\phi A(\phi), \qquad A_z(\phi) = \alpha\beta^{-1}\sin\phi A(\phi) \tag{6a}$$

The solutions (5) compelled to satisfy the condition (5a) are the nondispersive electromagnetic beams generalizing the Durnin scalar Bessel beams since for $A_j(\phi)$ constant the integral in (5) becomes the Bessel function $J_0(\alpha r)$ with $r = (x^2+y^2)^{1/2}$ while for instance the TEM modes (6a) depend on J_1 for $A(\phi)$ constant. The parameters α, β are real for $\omega > \omega_p$ while for $\omega < \omega_p$ and β real, α becomes complex giving a beam evanescent in the transverse direction: the electromagnetic beam is focused around Oz.

3 Bessel beam propagation in a stationary anisotropic plasma

We now consider a plasma in a steady magnetic field chosen to be spatially uniform and z-directed so that the plasma becomes anisotropic and characterized by a permittivity tensor $\varepsilon_{jk}(\omega)$. Then, we get instead of (1, 1a) the following equations for $\rho = 0$

$$\partial^j \partial_j E_k(\mathbf{x}, \omega) - \partial_k \partial^j E_j(\mathbf{x}, \omega) + \omega^2 c^{-2} \varepsilon^j_k(\omega) E_j(\mathbf{x}, \omega) = 0 \tag{7}$$

$$\partial^j D_j(\mathbf{x}, \omega) \equiv \varepsilon^{jk}(\omega) \partial_j E_k(\mathbf{x}, \omega) = 0. \tag{7a}$$

In this particular anisotropic medium, $\varepsilon_{jk}(\omega)$ has a tridiagonal symmetry

$$\varepsilon_{11}(\omega) = \varepsilon_{22}(\omega), \qquad \varepsilon_{12}(\omega) = -\varepsilon_{21}(\omega), \qquad \varepsilon_{13}(\omega) = \varepsilon_{31}(\omega) = \varepsilon_{23}(\omega) = \varepsilon_{32}(\omega) = 0 \tag{8}$$

and the non-null components have the expressions [13]

$$\varepsilon_{11}(\omega) = \varepsilon_0[1 + \omega_p^2(\omega_c^2 - \omega^2)^{-1}]$$
$$\varepsilon_{12}(\omega) = \mathrm{i}\,\varepsilon_0\, \omega_p^2\, \omega_c \left[\omega(\omega_c^2 - \omega^2)\right]^{-1}$$
$$\varepsilon_{33}(\omega) = \varepsilon_0(1 - \omega_p^2 \omega^{-2}) \tag{8a}$$

in which ω_p, ω_c are the plasma and the angular cyclotron frequencies.

Only circularly polarized plane waves are solutions of equations (7, 7a) when $\partial^j E_j(\mathbf{x}, \omega) = 0$, so we impose the weaker condition

$$\partial_x E_x(\mathbf{x}, \omega) + \partial_y E_y(\mathbf{x}, \omega) = 0 \tag{9}$$

and the equations (7) take the explicit form

$$\partial^j \partial_j E_x(\mathbf{x}, \omega) - \partial_x \partial_z E_z(\mathbf{x}, \omega) + \omega^2 c^{-2}[\varepsilon_{11}(\omega) E_x(\mathbf{x}, \omega) + \varepsilon_{12}(\omega) E_y(\mathbf{x}, \omega)] = 0$$

$$\partial^j \partial_j E_y(\mathbf{x}, \omega) - \partial_y \partial_z E_z(\mathbf{x}, \omega) + \omega^2 c^{-2}[-\varepsilon_{12}(\omega) E_x(\mathbf{x}, \omega) + \varepsilon_{11}(\omega) E_y(\mathbf{x}, \omega)] = 0$$

$$\partial^n \partial_n E_z(\mathbf{x}, \omega) + \omega^2 c^{-2} \varepsilon_{33}(\omega) E_z(\mathbf{x}, \omega) = 0 \qquad (\partial^n \partial_n = \partial_x^2 + \partial_y^2). \tag{10}$$

The condition (9) is fulfilled if we impose on E_x, E_y the constraints

$$\partial_x \partial_z E_z(\mathbf{x}, \omega) = \omega^2 c^{-2} \varepsilon_{12}(\omega) E_y(\mathbf{x}, \omega), \qquad \partial_y \partial_z E_z(\mathbf{x}, \omega) = -\omega^2 c^{-2} \varepsilon_{12}(\omega) E_x(\mathbf{x}, \omega) \tag{11}$$

so that we are left with the system of equations in which the subscripts 1, 2, correspond to x, y

$$\partial^j \partial_j E_n(\mathbf{x}, \omega) + \omega^2 c^{-2} \varepsilon_{11}(\omega) E_n(\mathbf{x}, \omega) = 0 \tag{12}$$

$$\partial^n \partial_n E_z(\mathbf{x}, \omega) + \omega^2 c^{-2} \varepsilon_{33}(\omega) E_z(\mathbf{x}, \omega) = 0 \tag{12a}$$

with according to (11)

$$E_y(\mathbf{x},\omega) = a(\omega)\partial_x\partial_z E_z(\mathbf{x},\omega),$$
$$E_x(\mathbf{x},\omega) = -a(\omega)\partial_y\partial_z E_z(\mathbf{x},\omega) \quad (13)$$

$$a(\omega) = \omega^{-2}c^2/\varepsilon_{12}(\omega). \quad (13a)$$

But the wave equation (12a) has the solutions

$$E_z(\mathbf{x},\omega) = \exp(-\mathrm{i}\beta z)\int_0^{2\pi} \mathrm{d}\phi A(\phi) \times \exp[\mathrm{i}\alpha_{33}(x\cos\phi + y\sin\phi)] \quad (14)$$

in which β is an arbitrary parameter and

$$\alpha_{33}^2 = \omega^2 c^{-2}\varepsilon_{33}(\omega). \quad (14a)$$

Substituting (14) into (13) gives the components $E_n(\mathbf{x},\omega)$ which are solution of (12) if

$$\beta^2 + \alpha_{33}^2 = \omega^2 c^{-2}\varepsilon_{11}(\omega) \quad (15)$$

and the solutions of the wave equations (10) are now fully determined. But we have still to check that the divergence equation (7) is satisfied, its explicit form is

$$\varepsilon_{11}(\omega)[\partial_x E_x(\mathbf{x},\omega) + \partial_y E_y(\mathbf{x},\omega)] + \varepsilon_{12}(\omega)[\partial_x E_y(\mathbf{x},\omega) - \partial_y E_x(\mathbf{x},\omega)] + \varepsilon_{33}(\omega)\partial_z E_z(\mathbf{x},\omega) = 0 \quad (16)$$

and substituting (9), (13) into (16) gives

$$a(\omega)\varepsilon_{12}(\omega)\partial^n\partial_n\partial_z E_z(\mathbf{x},\omega) + \varepsilon_{33}(\omega)\partial_z E_z(\mathbf{x},\omega) = 0 \quad (17)$$

but according to (14) $\partial^n\partial_n E_z(\mathbf{x},\omega) = -\alpha_{33}^2 E_z(\mathbf{x},\omega)$ so that (17) reduces to

$$-a(\omega)\varepsilon_{12}(\omega)\alpha_{33}^2 + \varepsilon_{33}(\omega) = 0 \quad (17a)$$

a relation satisfied according to the definitions (13a), (14a) of $a(\omega)$ and α_{33}^2.

Then, we get from (13), (14) the explicit form of the electromagnetic Bessel solutions in this tridiagonal anisotropic plasma

$$E_x(\mathbf{x},\omega) = -\beta\alpha_{33}\exp(-\mathrm{i}\beta z)\int_0^{2\pi} \mathrm{d}\phi A(\phi)\sin\phi \times \exp[\mathrm{i}\alpha_{33}(x\cos\phi + y\sin\phi)]$$

$$E_y(\mathbf{x},\omega) = \beta\alpha_{33}\exp(-\mathrm{i}\beta z)\int_0^{2\pi} \mathrm{d}\phi A(\phi)\cos\phi \times \exp[\mathrm{i}\alpha_{33}(x\cos\phi + y\sin\phi)]$$

$$E_z(\mathbf{x},\omega) = \exp(-\mathrm{i}\beta z)\int_0^{2\pi} \mathrm{d}\phi A(\phi) \times \exp[\mathrm{i}\alpha_{33}(x\cos\phi + y\sin\phi)] \quad (18)$$

with $a(\omega)$, α_{33}, β given by (13a), (14a) and (15). And for β real these beams propagate in the z-direction of the magnetic field. But, we get from (14a), (15) and (8a)

$$\beta = \omega c^{-1}[\varepsilon_{11}(\omega) - \varepsilon_{33}(\omega)]^{1/2} = \varepsilon_0 c^{-1}\omega_p\omega_c(\omega_c^2 - \omega^2)^{-1/2} \quad (19)$$

and propagation takes place only at frequencies $\omega < \omega_c$. A particular property of the solution (18) is that the off-diagonal components $\varepsilon_{12}(\omega)$, $\varepsilon_{21}(\omega)$ $\{= -\varepsilon_{11}(\omega)\}$ of the permittivity tensors intervene only to determine the field amplitudes.

Remark: these results hold valid for a plasma moving along Oz with a nonrelativistic constant velocity, we have just to change ω into its Doppler frequency shift ω_D.

4 Discussion

At a time when nanotechnologies are blossoming, it is natural to investigate the propagation of delta rather than of harmonic Bessel beams in plasmas and, we now sketch an approach to this problem. In space-time, the wave equation for the electric field is assuming $\partial^j E_j = 0$

$$\partial^j\partial_j\mathbf{E}(\mathbf{x},t) - c^{-2}\partial_t^2\int_{-\infty}^t \varepsilon(t-\tau)\mathbf{E}(\mathbf{x},\tau)\mathrm{d}\tau = 0 \quad (20)$$

with the Laplace transform

$$\partial^j\partial_j\mathbf{E}(\mathbf{x},p) - c^{-2}p^2\varepsilon(p)\mathbf{E}(\mathbf{x},p) = 0. \quad (20a)$$

We are interested in the propagation of delta pulses $\mathbf{E}(\mathbf{x},t) = \mathbf{E}(x,y)\delta(t - z/v_z)$ in which δ is the Dirac distribution so that

$$\mathbf{E}(\mathbf{x},p) = \mathbf{E}(x,y)\exp(-pz/v_z). \quad (21)$$

We consider a plasma with the permittivity in which ν is the collision frequency and ω_p the plasma frequency

$$\varepsilon(t) = \varepsilon_0[\delta(t) - \omega_p^2\nu^{-1}\exp(-\nu|t|)] \quad (22)$$

with the Laplace transform

$$\varepsilon(p) = \varepsilon_0[1 - \omega_p^2(\nu^2 + \nu p)^{-1}] \quad (22a)$$

(the Fourier transform of (22) is the expression (2) with ω^{-2} changed into $(\omega^2+\nu^2)^{-1}$).

Substituting (21) and (22a) into (20a) gives $(n = 1,2)$

$$[\partial^n\partial_n - p^2k^2(p)]\exp(-pz/v_z)\mathbf{E}(x,y) = 0,$$
$$k^2(p) = \varepsilon_0 c^{-2}[1 - \omega_p^2(\nu^2+\nu p)^{-1} - c^2 v_z^{-2}] \quad (23)$$

with the solutions

$$\mathbf{E}(\mathbf{x},p) = \exp(-pz/v_z)\int_0^{2\pi} \mathrm{d}\phi\mathbf{A}(\phi) \times \exp[-pk(p)(x\cos\phi + y\sin\phi)]. \quad (24)$$

But in this case, we have to perform numerically [14] the inverse Laplace transform of (24).

We have obtained in this work a new class of electromagnetic waves made of Bessel beams propagating in idealized plasmas and we have been careful to list the simplifying assumptions making possible these idealizations. This class is very rich, as shown by the relations (6a) in neutral isotropic media. On the contrary, no Bessel beam can propagate in anisotropic media when one of the following conditions is not fulfilled: uniform magnetic field B, constant density, propagation in the B-direction. Then,with respect to experiments in which electromagnetic beams are injected in a magnetically confined plasma, the absence of propagation in the direction transverse to B is rather restrictive. Now the wave equation (7) and the divergence equation (7a), this last one discussed at length in [15], hold valid in any plasma since they are a direct consequence of Maxwell's theory. But, an important question concerns the permittivity tensor $\varepsilon_{jk}(\omega)$ which condenses the physical properties of a medium needed from an electromagnetic point of view and, in spite of many works, it is sometimes difficult to get reliable approximations of $\varepsilon_{jk}(\omega)$. Now, concerning electromagnetic wave propagation, we have to make a clear distinction between the symmetry of this tensor and the expression of its components as a function of frequency. The second aspect is important to determine the range of frequencies inside which propagation is possible while its symmetry points out whether equations (7, 7a) may have solutions or not. We conjecture that the tridiagonal symmetry is the weakest symmetry making solutions of these equations available.

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